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On the univalence conditions for certain class of analytic functions

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Abstract
A univalence condition for certain class of analytic functions was discussed by D. Yang and S. Owa (Hokkaido Math. J. **32** (2003), 127 - 136). In the present paper, by discussing some subordination relation, a new univalence condition is deduced.

1 Introduction

Let $\mathcal{H}$ denote the class of functions $p(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n and a complex number a, let $\mathcal{H}[a, n]$ be the class of functions $p(z) \in \mathcal{H}$ of the form

$$p(z) = a + \sum_{k=n}^{\infty} a_k z^k.$$

Also, let $\mathcal{A}$ be the class of functions $f(z) \in \mathcal{H}$ which are normalized by $f(0) = f'(0) - 1 = 0$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$ is denoted by $\mathcal{S}$.
In 1972, Ozaki and Nunokawa [2] proved a univalence criterion for $f(z) \in \mathcal{A}$ as follows.

Lemma 1.1 *If $f(z) \in \mathcal{A}$ satisfies*

$$\left| \frac{z^2 f'(z)}{\left(f(z)\right)^2} - 1 \right| < 1 \qquad (z \in \mathbb{U}),$$

then $f(z)$ is univalent in $\mathbb{U}$, which means that $f(z) \in \mathcal{S}$.

Let $p(z)$ and $q(z)$ be members of the class $\mathcal{H}$. Then the function $p(z)$ is said to be subordinate to $q(z)$ in $\mathbb{U}$, written by $p(z) \prec q(z) \quad (z \in \mathbb{U})$, if there exists a function $w(z) \in \mathcal{H}$ with $w(0) = 0$, $|w(z)| < 1 \quad (z \in \mathbb{U})$, and such that $p(z) = q(w(z)) \quad (z \in \mathbb{U})$. From the definition of the subordinations, it is easy to show that $p(z) \prec q(z) \quad (z \in \mathbb{U})$ implies that

$$p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset q(\mathbb{U}). \tag{1.1}$$

In particular, if $q(z)$ is univalent in $\mathbb{U}$, then we see that $p(z) \prec q(z) \quad (z \in \mathbb{U})$ is equivalent to the condition (1.1) by considering the function

$$w(z) = q^{-1}(p(z)) \qquad (z \in \mathbb{U}).$$

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Let $\mathcal{T}(\lambda,\mu)$ denote the class of functions $f(z) \in \mathcal{A}$ which satisfy $\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U})$ and the inequality

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - \lambda z^2 \left(\frac{z}{f(z)}\right)'' - 1 \right| < \mu \qquad (z \in \mathbb{U}) \tag{1.2}$$

for some real number μ ($\mu > 0$) and for some complex number λ. Yang and Owa [4] discussed the univalency for $f(z) \in \mathcal{T}(\lambda,\mu)$ as follows.

Lemma 1.2 *Let λ be a complex number with $\operatorname{Re}\lambda \geqq 0$. Then the class $\mathcal{T}(\lambda,\mu)$ is a subclass of $\mathcal{S}$ for some real number μ with $0 < \mu \leqq |1+2\lambda|$.*

To obtain the assertion in Lemma 1.2, Yang and Owa [4] discussed the following subordination relation.

Lemma 1.3 *Let λ be a complex number with $\lambda \neq 0$ and $\operatorname{Re}\lambda \geqq 0$. If $p(z) \in \mathcal{H}[1,n]$ satisfies the following subordination*

$$p(z) + \lambda z p'(z) \prec 1 + \mu z \qquad (z \in \mathbb{U})$$

for some real number μ ($\mu > 0$), then

$$p(z) \prec 1 + \frac{\mu}{1+n\lambda} z \qquad (z \in \mathbb{U}).$$

In the present paper, we discuss the subordination relation in Lemma 1.3 for the case that $\operatorname{Re}\lambda$ is negative, and deduce an extension of the assertion in Lemma 1.2.

2 Preliminaries

In order to discuss our main results, we will make use of several lemmas.

A function $L(z,t)$ for $z \in \mathbb{U}$ and $t \geqq 0$ is said to be a subordination (or Loewner) chain if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geqq 0$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$, and

$$L(z,s) \prec L(z,t) \qquad (z \in \mathbb{U})$$

when $0 \leqq s \leqq t$ (Pommerenke [3] or Miller and Mocanu [1]). Pommerenke [3] derived a necessary and sufficient condition for $L(z,t)$ to be a subordination chain bellow.

Lemma 2.1 *The function $L(z,t) = \sum_{k=1}^{\infty} a_k(t) z^k$ with $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$ for $z \in \mathbb{U}$ and $t \geqq 0$ is a subordination chain if and only if*

$$\operatorname{Re}\left\{ \frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}} \right\} > 0$$

for $z \in \mathbb{U}$ *and* $t \geqq 0$.

For $0 < r_0 \leqq 1$, we let

$$\mathbb{U}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}, \quad \partial\mathbb{U}_{r_0} = \{z \in \mathbb{C} : |z| = r_0\}$$

and $\overline{\mathbb{U}_{r_0}} = \mathbb{U}_{r_0} \cup \partial\mathbb{U}_{r_0}$. In particular, we write $\mathbb{U}_1 = \mathbb{U}$.

Miller and Mocanu [1] derived the following lemma which is related to the subordination of two functions as follows.

Lemma 2.2 *Let* $p(z) \in \mathcal{H}[a,n]$ *with* $p(z) \not\equiv a$. *Also, let* $q(z)$ *be analytic and univalent on the closed unit disk* $\overline{\mathbb{U}}$ *except for at most one pole on* $\partial\mathbb{U}$ *with* $q(0) = a$. *If* $p(z)$ *is not subordinate to* $q(z)$ *in* $\mathbb{U}$, *then there exist two points* $z_0 \in \partial\mathbb{U}_r$ *with* $0 < r < 1$ *and* $\zeta_0 \in \partial\mathbb{U}$, *and a real number* k *with* $k \geqq n$ *for which* $p(\mathbb{U}_r) \subset q(\mathbb{U})$,

$$(i) \quad p(z_0) = q(\zeta_0)$$

and

$$(ii) \quad z_0 p'(z_0) = k\zeta_0 q'(\zeta_0).$$

This lemma plays a crucial role in developing the theory of differential subordinations.

3 Main results

By making use of Lemma 2.1 and Lemma 2.2, we first develop the assertion concerned with the differential subordinations bellow.

Theorem 3.1 *Let* n *be a positive integer, and let* λ *be a complex number with*

$$\text{(3.1)} \qquad \operatorname{Re}\lambda \leqq 0 \quad \text{and} \quad \left|\lambda + \frac{1}{2n}\right| > \frac{1}{2n}.$$

Also, let $q(z)$ *be analytic in* $\mathbb{U}$ *with* $q(0) = a$, $q'(0) \neq 0$ *and*

$$\text{(3.2)} \qquad \operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > -\frac{1}{n}\operatorname{Re}\left(\frac{1}{\lambda}\right) \qquad (z \in \mathbb{U}).$$

If $p(z) \in \mathcal{H}[a,n]$ *satisfies the following subordination*

$$\text{(3.3)} \qquad p(z) + \lambda z p'(z) \prec q(z) + \lambda n z q'(z) \qquad (z \in \mathbb{U}),$$

then $p(z) \prec q(z) \quad (z \in \mathbb{U})$.

Proof. Noting that $q'(0) \neq 0$ and $\operatorname{Re}\lambda \leqq 0$, it follows from the inequality (3.2) that the function $q(z)$ is convex univalent in $\mathbb{U}$. Moreover, if we set

$$\text{(3.4)} \qquad h(z) = q(z) + \lambda n z q'(z) \qquad (z \in \mathbb{U}),$$

then, from the inequality (3.2), we find that

$$\operatorname{Re}\left(\frac{h'(z)}{\lambda q'(z)}\right) = \operatorname{Re}\left\{\frac{1}{\lambda} + n\left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0 \quad (z \in \mathbb{U}). \tag{3.5}$$

Since the function $\lambda q(z)$ is convex univalent in $\mathbb{U}$, the inequality (3.5) shows that the function $h(z)$ is close-to-convex in $\mathbb{U}$, which implies that $h(z)$ is univalent in $\mathbb{U}$ (cf. [1]).

If we define the function $L(z,t)$ by

$$L(z,t) = q(z) - a + (n+t)\lambda z q'(z) \tag{3.6}$$

for $z \in \mathbb{U}$ and $t \geqq 0$, then the function $L(z,t) = a_1(t)z + \cdots$ is analytic in $\mathbb{U}$ for all $t \geqq 0$, and continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$. Since $q'(0) \neq 0$, it is clear that

$$a_1(t) = \left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = \{1 + \lambda(n+t)\}q'(0) \neq 0 \qquad (t \geqq 0)$$

and

$$\lim_{t\to\infty} |a_1(t)| = \lim_{t\to\infty} \left|\{1 + \lambda(n+t)\}q'(0)\right| = \infty.$$

From the inequality (3.2), we obtain

$$\begin{aligned}\operatorname{Re}\left\{\frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}}\right\} &= \operatorname{Re}\left(\frac{1}{\lambda}\right) + (n+t)\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) \\ &\geqq \operatorname{Re}\left(\frac{1}{\lambda}\right) + n\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > 0\end{aligned}$$

for $z \in \mathbb{U}$ and $t \geqq 0$. Then by Lemma 2.1, $L(z,t)$ is subordination chain, and we have $L(z,s) \prec L(z,t) \quad (z \in \mathbb{U})$, when $0 \leqq s \leqq t$. We now set $\hat{L}(z,t) = L(z,t) + a$. From (3.4) and (3.6), we obtain $h(z) = \hat{L}(z,0) \prec \hat{L}(z,t)$ for $z \in \mathbb{U}$ and $t \geqq 0$. Thus, we see that

$$\hat{L}(\zeta,t) \notin h(\mathbb{U}) \tag{3.7}$$

for $|\zeta| = 1$ and $t \geqq 0$.

Without loss of generality, we can assume that $q(z)$ is univalent on the closed unit disk $\overline{\mathbb{U}}$. If we assume that $p(z)$ is not subordinate to $q(z)$ in $\mathbb{U}$, then by Lemma 2.1, there exist two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$, and a real number k with $k \geqq n$ such that $p(z_0) = q(\zeta_0)$ and $z_0 p'(z_0) = k\zeta_0 q'(\zeta_0)$. Then from (3.6) and (3.7), we have

$$p(z_0) + \lambda z_0 p'(z_0) = q(\zeta_0) + \lambda k \zeta_0 q'(\zeta_0) = \hat{L}(\zeta_0, k-n) \notin h(\mathbb{U}),$$

where $z_0 \in \mathbb{U}$, $|\zeta_0| = 1$ and $k \geqq n$. This contradicts the assumption (3.3) of the theorem, and hence we must have $p(z) \prec q(z) \quad (z \in \mathbb{U})$. This completes the proof of Theorem 3.1. □

Let us consider the function $q(z)$ given by

$$q(z) = 1 + \frac{\mu}{1+n\lambda}z \qquad (z \in \mathbb{U})$$

for some real number μ ($\mu > 0$) and for some complex number λ with the condition (3.1). Then, it is easy to see that

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) = 1 > -\frac{1}{n}\operatorname{Re}\left(\frac{1}{\lambda}\right) \qquad (z \in \mathbb{U})$$

and

$$q(z) + \lambda nzq'(z) = 1 + \mu z.$$

Hence by Theorem 3.1, we obtain

Theorem 3.2 *Let n be a positive integer, and let λ be a complex number with the condition (3.1). If $p(z) \in \mathcal{H}[1,n]$ satisfies the following subordination*

$$p(z) + \lambda zp'(z) \prec 1 + \mu z \qquad (z \in \mathbb{U})$$

for some real number μ ($\mu > 0$), then

$$p(z) \prec 1 + \frac{\mu}{1+n\lambda}z \qquad (z \in \mathbb{U}).$$

By combining Lemma 1.3 and Theorem 3.2, we find the following subordination assertion.

Theorem 3.3 *Let n be a positive integer, and let λ be a complex number with the inequality*

$$\left|\lambda + \frac{1}{2n}\right| > \frac{1}{2n}. \tag{3.8}$$

If $p(z) \in \mathcal{H}[1,n]$ satisfies the following subordination

$$p(z) + \lambda zp'(z) \prec 1 + \mu z \qquad (z \in \mathbb{U})$$

for some real number μ ($\mu > 0$), then

$$p(z) \prec 1 + \frac{\mu}{1+n\lambda}z \qquad (z \in \mathbb{U}).$$

For the function $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k \in \mathcal{A}$, we now set

$$p(z) = \frac{z^2 f'(z)}{(f(z))^2} = 1 + (a_3 - a_2{}^2)z^2 + \cdots \qquad (z \in \mathbb{U})$$

in Theorem 3.3. Noting that $n = 2$, we derive the following corollary.

Corollary 3.4 *Let λ be a complex number with $\left|\lambda + \frac{1}{4}\right| > \frac{1}{4}$. If $f(z) \in \mathcal{A}$ satisfies*

$$\frac{z^2 f'(z)}{(f(z))^2} - \lambda z^2 \left(\frac{z}{f(z)}\right)'' \prec 1 + \mu z \qquad (z \in \mathbb{U})$$

for some real number μ ($\mu > 0$), then

$$\frac{z^2 f'(z)}{\left(f(z)\right)^2} \prec 1 + \frac{\mu}{1+2\lambda} z \qquad (z \in \mathbb{U}).$$

From Corollary 3.4, we find that if $f(z) \in \mathcal{A}$ satisfies the inequality (1.2), then

$$\left| \frac{z^2 f'(z)}{\left(f(z)\right)^2} - 1 \right| < \frac{\mu}{|1+2\lambda|} \qquad (z \in \mathbb{U}) \tag{3.9}$$

for some real number μ ($\mu > 0$) and for some complex number λ with the inequality (3.8). According to Lemma 1.1, the inequality (3.9) shows that $f(z) \in \mathcal{S}$ if $0 < \mu \leqq |1+2\lambda|$. Thus, we obtain the following assertion.

Theorem 3.5 *Let λ be a complex number with the inequality* (3.8). *Then the class $\mathcal{T}(\lambda, \mu)$ is a subclass of $\mathcal{S}$ for some real number μ with $0 < \mu \leqq |1 + 2\lambda|$.*

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